M.Sc. (I) (Mathematics) (Old and CBCS Pattern) Sem-II

0168 / PSCMTHT06 - Paper-I : Algebra-II

P. Pages : 2 **GUG/S/19/11211**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) If R is a unique factorization Domain, then prove that the factorization of any element in R as a finite product of irreducible factors is unique to with in order and unit factors. **10**

b) Let R be unique factorization domain and $a, b \in R$. Then prove that there exists a greatest common divisor of a and b that is uniquely determined to within an arbitrary unit factor. **10**

OR

c) Show that: **10**
Every principal ideal domains is a unique factorization domain, but a unique factorization domain is not necessarily a principal ideal domain.

d) If $f(X), g(X) \in R[X]$, then prove that $C(f_g) = C(f)\ C(g)$. In particular the product of two primitive polynomials is primitive. **10**

UNIT – II

2. a) Let $F(X) = a_0 + a_1 X + \dots\dots\dots + a_n X^n \in z[x]$ If there is a prime P such that $p^2 \nmid a_0$, p / a_0, $p / a_1, \dots\dots\dots, p / a_{n-1}$, $p / \nmid a_n$ then prove that F(x) is irreducible over Q. **10**

b) Let E and F be fields and let $\sigma : F \to E$ be an embedding of F into E. Then prove that there exists a field K such that F is a sub field of K and σ can be extended an isomorphism of K on to E. **10**

OR

c) Let E be an extension field of F and let $u \in E$ be algebraic over F. Let $p(x) \in F[x]$ be a polynomial of the least degree such that $P(u) = 0$. Then prove that **10**

i) P(x) is irreducible over F.
ii) If $g(X) \in F[x]$ is such that g(u) = 0, then P(x) / g(x)
iii) There is exactly one monic polynomial $P(x) \in F[x]$ of least degree such that p(u) = 0.

d) Let K be a Spliting field of the polynomial $f(x) \in F[x]$ over a field F. If E is another spliting field of f(x) over F. Then prove that there exist an isomorphism $\sigma : E \to K$ that is identity on F. **10**

UNIT – III

3. a) Show that any finite field F with p^n elements is the spliting field of $x^{p^n} - x \in F_p[x]$. consequently any two finite fields with p^n elements are isomorphic. **10**

b) Let E be an extension of a field F, and let $\alpha \in E$ be algebraic over F. Then prove that α is separable over F iff $F(\alpha)$ is a separable extension of F. **10**

OR

c) If $f(x) \in F[x]$ has r distinct roots in its splitting field E over F, then prove that the Galois group $G\left(E/F\right)$ of F(x) is a subgroup of the symmetric group S_r. **10**

d) Show that every polynomial $f(x) \in C[x]$ factors into linear factors in $C[x]$. **10**

UNIT – IV

4. a) Let F be a field and let U be a finite subgroup of the multiplicative group $F^* = F - \{O\}$. Then prove that U is cyclic. In particular the roots of $x^n - 1 \in F[x]$ form a cyclic group. **10**

b) Let E be a finite extension of F. Suppose $f : G \to E^*$, $E^* = E - \{O\}$ has the property that $f(\sigma n) = \sigma(f(\eta)) \cdot f(\sigma)$ for all $\sigma, n \in G$. Then prove that there exists $\alpha \in E^*$ such that $f(\sigma) = \sigma(\alpha^{-1})\alpha$ for all $\sigma \in G$. **10**

OR

c) Let F(x) be a polynomial over a field F with no multiple roots. Then prove that f(x) is irreducible over F if and only if the Galois group G of F(x) is isomorphic to a transitive permutation group. **10**

d) Let $f(x) \in Q[x]$ be a monic irreducible polynomial over Q of degree P, where P is prime. If F(x) has exactly two non real roots in C. then prove that the Galois group of f(x) is isomorphic to S_P. **10**

5. a) Show that every Euclidean domain is a principal ideal domain. **5**

b) Let $F = z/(2)$. Then show that the spliting field of $x^3 + x^2 + 1 \in F[x]$ is a finite field with eight elements. **5**

c) Let F be field of characteristic $\neq 2$ Let $x^2 - a \in F[x]$ be an irreducible polynomial over F. Then prove that its Galois group is of order 2. **5**

d) Show that the Galois group of $x^4 + x^2 + 1$ is the same as that of $x^6 - 1$ and is of order 2. **5**

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